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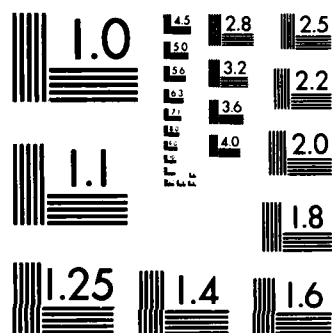
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The focus of this research is the filtering jump processes. To investigate the filtering of manifold-valued processes, their approximation by random walks and Markov chains was studied. The object was to approximate a signal process by a finite-state jump process for which a finite-dimensional filter is available. Four papers were published during the past year, including "The existence of smooth densities for the prediction, filtering and smoothing problems" and "The partially observed stochastic minimum principle".

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Prediction Filtering and Smoothing Problems**

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# The Existence of Smooth Densities for the Prediction Filtering and Smoothing Problems

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## ABSTRACT

Using a simple martingale representation result a conditional version of the Malliavin calculus is developed. Under Hörmander's conditions on the coefficient vector fields we show the filtering, smoothing and prediction problems have  $C^\infty$  density solutions.

*and a superscript is T*

1. INTRODUCTION. Following Malliavin's remarkable work [6] there have been other treatments of the Malliavin calculus, including those of Bismut [1], Stroock [8] and Norris [7]. A particularly readable account can be found in the paper of Zakai [9]. In [2] Bismut and Michel developed a conditional version of the Malliavin calculus to show the existence of a conditional density in filtering and smoothing problems. Using a simple and natural expression for the integrand in a stochastic integral the authors [4] have been able to give an elementary proof of the existence of a density for a diffusion,

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under Hörmander's conditions for the coefficient vector fields. The homogeneous chaos expansion of the random variable is also obtained in [4]. The objective of this paper is to present a conditional version of the results of [4] and, following the exposition of Zakai, simplify some of the results of Bismut and Michel.

In this paper the following system of stochastic differential equations is considered:

$$dx = X_0(x, y)dt + X_i(x, y)dw^i + \tilde{X}_j(x, y)dB^j + \tilde{X}_j(x, y)h^j(x, y)dt.$$

$$dy = Y_0(y)dt + Y_j(y)dB^j + Y_j(y)h^j(x, y)dt.$$

Here  $w = (w^1, \dots, w^n)$  and  $B = (B^1, \dots, B^n)$  are independent Brownian motions. The process  $x$  represents the unobserved signal process, while  $y$  represents the observation process. If  $\{Y_t\}$  is the right continuous, complete filtration generated by  $\{y_t\}$  then the filtering problem discusses  $E[x_t|Y_t]$ , the prediction problem discusses  $E[x_t|Y_s]$  when  $s \leq t$ , and the smoothing problem discusses  $E[x_t|Y_s]$  when  $s \geq t$ .

Using the simple martingale representation result of [4] a conditional version of the Malliavin calculus is developed in section 4. Suppose  $T \geq t$  and let  $c$  be any smooth function on  $R^d$  with bounded derivatives of all orders. In section 5 we show that if the inverse of the conditional Malliavin matrix  $M$  belongs to  $L^p(\Omega)$  for all  $p, 1 \leq p < \infty$ , then

$$|E[\frac{\partial^\alpha c}{\partial x^\alpha}(x_t)|Y_T]| \leq K(y) \sup_{x \in R^d} |c(x)|$$

for all multi-indices  $\alpha = (\alpha_1, \dots, \alpha_d)$  where  $K(y)$  is a  $Y_T$ -measurable random variable which is finite a.s. The delicate and technical sufficient conditions for the integrability of  $M^{-p}$  are not discussed in this paper.

This inequality, using simple Fourier analysis, implies that the random variable  $x_t$  has almost surely a conditional density given  $Y_T$ , which is infinitely differentiable. Using Jensen's inequality we can immediately deduce

$$|E[\frac{\partial^\alpha c}{\partial x^\alpha}(x_t)|Y_s]| \leq K'(y) \sup_{x \in \mathbb{R}^d} |c(x)|$$

where  $s \geq t$  or  $s \leq t$ . Therefore, the smoothing, filtering and prediction problems for  $x_t$ , given  $Y_s$ , have, almost surely, smooth conditional density solutions.

## 2. STOCHASTIC FLOWS

We recall in this section the properties of stochastic flows, and in particular those relating to 'lower triangular' systems obtained by Norris [7]. Let  $w_t = (w_t^1, \dots, w_t^n)$ ,  $t \geq 0$ , be an  $n$ -dimensional Brownian motion on  $(\Omega, F, P)$ . Write  $\{F_t\}$  for the right continuous, complete filtration generated by  $w$ . Suppose  $X_0, X_1, \dots, X_m$  are smooth vector fields on  $[0, \infty) \times \mathbb{R}^d$ , all of whose derivatives are bounded. Then from Bismut [1], or Carverhill and Elworthy [3], we quote the following result:

**THEOREM 2.1.** *There is a map  $\xi : \Omega \times [0, \infty) \times [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  such that*

*i) for  $0 \leq s \leq t$  and  $x \in \mathbb{R}^d$   $\xi_{s,t}(x)$  is the essentially unique solution of the stochastic differential equation*

$$d\xi_{s,t}(x) = X_0(t, \xi_{s,t}(x))dt + X_i(t, \xi_{s,t}(x))dw_t^i \quad (2.1)$$

*with  $\xi_{s,s}(x) = x$ .*

*(Note the Einstein summation convention is used).*

*ii) for each  $\omega, s, t$  the map  $\xi_{s,t}(\cdot)$  is  $C^\infty$  on  $\mathbb{R}^d$  with a first derivative, the Jacobian,  $\frac{\partial \xi_{s,t}}{\partial x} = D_{s,t}$ , which satisfies*

$$dD_{s,t} = \frac{\partial X_0}{\partial \xi}(t, \xi_{s,t}(x))D_{s,t}dt + \frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x))D_{s,t}dw_t^i \quad (2.2)$$

*with initial condition  $D_{s,s} = I$ , the  $d \times d$  identity matrix.*

**REMARKS 2.2.** Note that (2.2) is obtained formally by differentiating (2.1). In fact equations for higher derivatives  $\frac{\partial^n \xi}{\partial x^n}$  are obtained by further differentiation. However, if we consider the enlarged system given by (2.1) and (2.2) the coefficients are not



bounded, because of the linear appearance of  $D_{s,t}$  on the right of (2.2). However, Norris [7] has extended the results of Theorem 2.1 to such systems. To state Norris' results we first define a class of 'lower triangular' coefficients.

DEFINITION 2.3. For positive integers  $\alpha, d, d_1, \dots, d_k$  write  $S_\alpha(d_1, \dots, d_k)$  for the set of  $X \in C^\infty(R^d, R^d)$  of the form

$$X(x) = \begin{pmatrix} X^{(1)}(x^1) \\ X^{(2)}(x^1, x^2) \\ \vdots \\ X^{(k)}(x^1, x^2, \dots, x^k) \end{pmatrix} \text{ for } x = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^k \end{pmatrix} \quad (2.3)$$

where  $R^d$  is identified with  $R^{d_1} \times \dots \times R^{d_k}$ ,  $x^j \in R^{d_j}$  and the  $X$  satisfy

$$\begin{aligned} \|X\|_{S(\alpha, N)} &= \sup_{x \in R^d} \left( \sup_{0 \leq n \leq N} \frac{|D^n X(x)|}{(1 + |x|^\alpha)} \vee \sup_{1 \leq j \leq k} |D_j X^{(j)}(x)| \right) \\ &< \infty \quad \text{for all positive integers } N. \end{aligned} \quad (2.4)$$

Write  $S(d_1, \dots, d_k) = \bigcup_\alpha S_\alpha(d_1, \dots, d_k)$ .

REMARKS 2.4. Note equations (2.1) and (2.2) can be considered as a single system whose coefficients are not bounded, but are in  $S(d, d^2)$ . The final supremum on the right of (2.4) implies the first derivatives of  $X^{(1)}$  are bounded, as are the first derivatives  $D_j$  in the 'new' variable  $x^j$  of  $X^{(j)}(x^1, \dots, x^j)$ . This means  $X^{(j)}$  is allowed linear growth in  $x^j$ , a situation illustrated in (2.2). We quote from Norris the following extension of Theorem 2.1.

THEOREM 2.5. Let  $X_0, X_1, \dots, X_m \in S_\alpha(d_1, \dots, d_k)$ . Then there is a map  $\phi : \Omega \times [0, \infty) \times [0, \infty) \times R^d \rightarrow R^d$  such that

i) for  $0 \leq s \leq t$  and  $x \in R^d$   $\phi(\omega, s, t, x)$  is the essentially unique solution of the stochastic differential equation

$$dx_t = X_0(x_t)dt + X_i(x_t)dw_t^i \quad (2.5)$$

with  $x_s = x$ .

ii) for each  $\omega, s, t$  the map  $\phi(\omega, s, t, x)$  is  $C^\infty$  in  $x$  with derivatives of all orders satisfying stochastic differential equations obtained from (2.5) by formal differentiation.

iii)

$$\begin{aligned} \sup_{|x| \leq R} E \left[ \sup_{s \leq u \leq t} |D^N \phi(\omega, s, u, x)|^p \right] \\ \leq C(p, s, t, R, N, d_1, \dots, d_k, \alpha, \|X_0\|_{S(\alpha, N)}, \dots, \|X_n\|_{S(\alpha, N)}). \end{aligned} \quad (2.6)$$

REMARKS 2.6. Norris proves Theorem 2.5 by induction on  $j$ . Write (2.5) as a system of stochastic differential equations for  $j = 1, \dots, k$

$$\begin{aligned} dx_t^j &= X_0^{(j)}(x_t^1, \dots, x_t^j)dt + X_i^{(j)}(x_t^1, \dots, x_t^j)dw_t^i \\ x_s^j &= x^j \in R^{d_j}. \end{aligned} \quad (2.7)$$

Suppose the result is true for  $1 \leq j \leq j-1$  and write  $\tilde{X}_i^{(j)}(\omega, s, t, x^j) = X_i^{(j)}(x_t^1(\omega), \dots, x_t^{j-1}(\omega), x^j)$ . Then (2.7) can be written in the form

$$dx_t^j = \tilde{X}_0(s, t, x_t^j)dt + \tilde{X}_i(s, t, x_t^j)dw_t^i$$

and Theorem 2.1 applied. The difficult step is establishing the result for

$j = 1$ . However, this follows by a stopping time argument, which is essentially the

method by Bismut [1]. Using the notation of Theorem 2.1 consider the process  $V$  defined by

$$\begin{aligned} dV_{s,t} = & -V_{s,t} \left( \frac{\partial X_0}{\partial \xi}(t, \xi_{s,t}(x)) - \sum_{i=1}^n \left( \frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x)) \right)^2 dt \right. \\ & \left. - V_{s,t} \frac{\partial X_i}{\partial \xi}(t, \xi_{s,t}(x)) dw_t^i, \right. \end{aligned} \quad (2.8)$$

with  $V_{s,s} = I$ . Then by applying the Ito rule we see  $d(D_{s,t} V_{s,t}) = 0$ , while  $D_{s,s} V_{s,s} = I$ , the  $d \times d$  identity matrix. Therefore,  $V_{s,t} = D_{s,t}^{-1}$ . By applying Theorem 2.5(iii) to the system given by equations (2.1), (2.2) and (2.8) we have  $|D_{s,t}^*| = \sup_{s \leq u \leq t} |D_{s,u}|$  and  $|V_{s,t}^*| = \sup_{s \leq u \leq t} |V_{s,u}|$  are in  $L^p(\Omega)$  for all  $p < \infty$ . Finally, for  $0 \leq s \leq t$ , recall, by the uniqueness of the solution of (2.1):

$$\xi_{0,t}(x_0) = \xi_{s,t}(\xi_{0,s}(x_0)) = \xi_{s,t}(x), \text{ if } x = \xi_{0,s}(x_0). \quad (2.9)$$

Differentiating (2.9):

$$D_{s,t} = D_{s,t} D_{0,s} \quad (2.10)$$

and

$$V_{0,t} = V_{0,s} V_{s,t}. \quad (2.11)$$

### 3. MARTINGALE REPRESENTATION.

Consider a stochastic differential system with coefficients in some set  $S$ , as discussed in Theorem 2.5, and let  $\xi_{0,t}(x,0)$  be its stochastic flow solution. For some  $T > 0$  consider a real valued differentiable function  $c$  for which the random variable  $c(\xi_{0,T}(x_0))$  and the components of the gradient  $c_\xi(\xi_{0,T}(x_0))$  are integrable. Let  $M_t$  be the right continuous version of the martingale

$$E[c(\xi_{0,T}(x_0))|F_t].$$

Then we have the following representation result, (see [4]), whose proof we give for completeness.

**THEOREM 3.1.** For  $0 \leq t \leq T$ ,  $M_T = E[c(\xi_{0,T}(x_0))] + \int_0^t \gamma_i(s)dw_s^i$  where  $\gamma_i(s) = E[c_\xi(\xi_{0,T}(x_0))D_{0,T}|F_s]D_{0,s}^{-1}X_i(s, \xi_{0,s}(x_0))$ .

**PROOF.** It is well known that  $M_t$  has a representation

$$M_t = M_0 + \int_0^t \gamma_i(s)dw_s^i \quad (3.1)$$

for some predictable integrands  $\gamma_i$ . Because the process  $\xi_{0,T}(x_0)$  is Markov

$$\begin{aligned} M_t &= E[c(\xi_{0,T}(x_0))|F_t] \\ &= E[c(\xi_{t,T}(x))|F_t] \\ &= E_t[c(\xi_{t,T}(x))] \\ &= V(t, x), \text{ say, where } x = \xi_{0,t}(x_0). \end{aligned} \quad (3.2)$$

By Theorem 2.5 and the chain rule  $c(\xi_{t,T}(x))$  is differentiable, in fact smooth, in  $x$ . The differentiability of  $E_{t,x}[c(\xi_{t,T}(x))]$  in  $t$  can be established by writing the backward equation for  $\xi_{t,T}(x)$ , as in Kunita [5]. Consequently, applying the Ito rule to  $V(t, x)$ , with  $x = \xi_{0,T}(x_0)$  we have

$$\begin{aligned} V(t, \xi_{0,t}(x_0)) &= V(0, x_0) + \int_0^t \left( \frac{\partial V}{\partial s} + LV \right) ds \\ &\quad + \int_0^t \frac{\partial V}{\partial x}(s, \xi_{0,s}(x_0)) X_i(s, \xi_{0,s}(x_0)) dw_s^i \end{aligned} \quad (3.3)$$

where  $L = \sum_{i=1}^d X_0^i \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d \left( \sum_{k=1}^m X_k^i X_k^j \right) \frac{\partial^2}{\partial x_i \partial x_j}$ . By the uniqueness of the decomposition of special semimartingales, comparing (3.1) and (3.3), we must have, (as is well known),

$$\frac{\partial V}{\partial s} + LV = 0$$

and  $\gamma_i(s) = \frac{\partial V}{\partial x}(s, \xi_{0,s}(x_0)) X_i(s, \xi_{0,s}(x_0))$ . From (3.2)  $\frac{\partial V}{\partial x} = E[c_\xi(\xi_{s,T}(x)) D_{s,T} | F_s]$  so by (2.10)  $\gamma_i(s) = E[c_\xi(\xi_{0,T}(x_0)) D_{0,T} | F_s] D_{0,s}^{-1} X_i(s, \xi_{0,s}(x_0))$ .

**COROLLARY 3.2.** *The result extends immediately to vector (or matrix) functions  $c$*

**COROLLARY 3.3.** *Note in particular*

$$c(\xi_{0,T}(x_0)) = E[c(\xi_{0,T}(x_0))] + \int_0^T E[c_\xi(\xi_{0,T}(x_0)) D_{0,T} | F_s] D_{0,s}^{-1} X_i(s, \xi_{0,s}(x_0)) dw_s^i \quad (3.4)$$

**LEMMA 3.4.**  *$F_t$  is generated by the set of stochastic integrals of the form  $\int_0^t \gamma_i(s, w_s) dw_s^i$ , where the integrands  $\gamma_i$  are smooth functions of  $s$  and  $w_s$  at time  $s$ , with bounded derivatives of all orders.*

PROOF.  $\sigma\{w_t\}$  is generated by  $g(w_t)$  for  $g \in C_b^\infty(R^d)$ . If we apply Theorem 3.1 to the process  $w_t$ , so  $w_t = x + (w_t - w_s)$  where  $x = w_s$ , the Jacobian is the identity  $I$  and

$$E[g_w(w_t)|F_s] = E_{s,x}[g_w(w_t)] = \gamma(w_s)$$

where  $\gamma(w_s) = (\gamma_1(w_s), \dots, \gamma_m(w_s)) = E_{s,w_s}[(g_{w^1}(w_s), \dots, g_{w^m}(w_s))]$ . Therefore,  $g(w_t) = E[g(w_t)] + \int_0^t \gamma_i(w_s)dw_s^i$  where the  $\gamma_i \in C_b^\infty(R^d)$ . Consequently  $\sigma(w_t)$  is generated by stochastic integrals of this form. Allowing the integrands to depend on  $s$  we that  $F_t$ , which is generated by  $w_s$ , for  $s \leq t$ , is generated by stochastic integrals of the form

$$\int_0^t \gamma_i(s, w_s)dw_s^i$$

where  $\gamma_i \in C_b^\infty([0, \infty) \times R^n)$ .

REMARKS 3.5. So far we have considered an  $n$ -dimensional Brownian motion  $w = (w^1, \dots, w^m)$  and a state vector  $x \in R^d$ . Consider now a larger system: suppose  $B = (B^1, \dots, B^n)$  is an  $n$ -dimensional Brownian motion, defined on a probability space  $(\tilde{\Omega}, \tilde{F}, \tilde{P})$ , which is independent of  $w$ . Write  $\{\tilde{F}_t\}$  for the right continuous, complete filtration generated by  $B$ , and  $\{G_t\}$  for the right continuous, complete filtration on  $\Omega \times \tilde{\Omega}$  generated by  $F_t \times \tilde{F}_t$ . Consider a second state vector  $y \in R^p$  and a stochastic differential system defined on  $(\Omega \times \tilde{\Omega}, F \times \tilde{F}, P \times \tilde{P})$  by the equations

$$\begin{aligned} dx_t &= X_0(x_t, y_t)dt + X_i(x_t, y_t)dw_t^i + \tilde{X}_j(x_t, y_t)d\tilde{B}_t^j \\ dy_t &= Y_0(y_t)dt + Y_j(y_t)d\tilde{B}_t^j \end{aligned} \quad (3.5)$$

with  $(x(0), y(0)) = (x_0, y_0) \in R^d \times R^p$ . We shall suppose the coefficient vector fields  $X_0, \dots, X_m, Y_0, \dots, Y_n$  are such that the coefficients of (3.5) belong to the space  $S$ ,

so that Theorem 2.5 can be applied. Note that in (3.5) the process  $y$  is not influenced by the process  $x$ .

NOTATION 3.6. Suppose  $(x, y) \in R^d \times R^p$  is the state of the system (3.5) at time  $s$ . We shall denote the solution flow of (3.5) for  $t \geq s$  by the map

$$(x, y) \rightarrow (x_{s,t}(x, y), y_{s,t}(y))$$

The Jacobian of this map looks like

$$\frac{\partial(x_{s,t}(x, y), y_{s,t}(y))}{\partial(x, y)} = \begin{pmatrix} \frac{\partial(x_{s,t}(x, y))}{\partial x} & \frac{\partial(x_{s,t}(x, y))}{\partial y} \\ 0 & \frac{\partial(y_{s,t}(y))}{\partial y} \end{pmatrix}. \quad (3.6)$$

Write  $D_{s,t}(x, y)$  for the 'partial' Jacobian  $\frac{\partial(x_{s,t}(x, y))}{\partial x}$ . The existence of the large Jacobian, and, therefore, of its components, including  $D_{s,t}$ , is given by Theorem 2.5.

As in [2], we now introduce a new measure on  $(\Omega \times \tilde{\Omega}, F \times \tilde{F})$  by a Girsanov change of density.

NOTATION 3.7. Suppose  $h(x, y) = (h^1(x, y), \dots, h^n(x, y))$  is a smooth function in  $C^\infty(R^{d+p}, R^n)$  with bounded derivatives of all orders. Define the real valued process  $L$  on  $\Omega \times \tilde{\Omega} \times [0, \infty)^2 \times R^d \times R^p$  by

$$L_{s,t}(x, y) = \exp \left\{ \int_s^t h^j(x_{s,u}(x, y), y_{s,u}(y)) dB_u^j - \frac{1}{2} \sum_{j=1}^n \int_s^t h^j(x_{s,u}(x, y), y_{s,u}(y))^2 du \right\}.$$

Then

$$dL_{s,t}(x, y) = L_{s,t}(x, y) (h^j(x_{s,t}(x, y), y_{s,t}(x, y)) dB_t^j) \quad (3.7)$$

with  $L_{s,s}(x,y) = 1$ , so  $L$  is a  $\{G_t\}$  martingale. Furthermore,  $L_{0,t}^* = \sup_{u \leq t} L_{0,u}$  is in every space  $L^p(\Omega)$ ,  $1 \leq p < \infty$ . Because  $h$  is bounded we also have that  $(L_{0,t}^{-1})^* = \sup_{u \leq t} (L_{0,u}^{-1})$  is in every  $L^p(\Omega)$ ,  $1 \leq p < \infty$ . We could consider the flow given by the combined system (3.5) and (3.7). However, for the moment note that for  $0 \leq s \leq t$

$$L_{0,t}(x_0, y_0) = L_{0,s}(x_0, y_0) L_{s,t}(x, y) \quad (3.8)$$

so writing  $L = L_{0,s}(x_0, y_0)$  we have

$$\frac{\partial L_{0,t}}{\partial L} = L_{s,t}(x, y)$$

and

$$\frac{\partial L_{0,t}}{\partial x_0} = \frac{\partial L_{0,s}}{\partial x_0} L_{s,t}(x, y) + L_{0,s} \frac{\partial L_{s,t}}{\partial x} D_{0,s} \quad (3.9)$$

with a similar equation for  $\frac{\partial L_{0,t}}{\partial y_0}$ .

DEFINITION 3.8. Define a measure  $P_h$  on  $(\Omega \times \tilde{\Omega}, F \times \tilde{F})$  such that its restriction to  $G_t$  is given by

$$dP_h(\omega, \tilde{\omega}) = L_{0,t}(x_0, y_0) dP(\omega) \times d\tilde{P}(\tilde{\omega}).$$

Then Girsanov's theorem states:

THEOREM 3.9. Under  $P_h$  the process  $B'$  is an  $n$ -dimensional Brownian motion independent of  $w$ , where

$$B'_t = B_t - \int_0^t h(x_{0,s}, y_{0,s}) ds.$$



Therefore, under the measure  $P_h$  the process  $(x_{s,t}, y_{s,t})$  is the solution of the stochastic differential equation

$$\begin{aligned} dx_{s,t} &= X_0(x_{s,t}, y_{s,t})dt + X_i(x_{s,t}, y_{s,t})dw_t^i + \tilde{X}_j(x_{s,t}, y_{s,t})dB_t^{ij} \\ &\quad + \tilde{X}_j(x_{s,t}, y_{s,t})h^j(x_{s,t}, y_{s,t})dt \\ dy_{s,t} &= Y_0(y_{s,t})dt + Y_j(y_{s,t})dB_t^{ij} + Y_j(y_{s,t})h^j(x_{s,t}, y_{s,t})dt \end{aligned} \quad (3.10)$$

with  $(x_{s,s}, y_{s,s}) = (x, y) \in R^d \times R^p$ .

REMARKS 3.10. The system (3.9) provides a natural setting in which to discuss filtering, smoothing or prediction problems. The process  $x_t$  represents a signal which is not observed directly. Instead one observes the process  $y_t$  which is influenced by  $x_t$  through the process  $h(x_t, y_t)$ . Write  $\{Y_t\}$  for the right continuous, complete filtration generated by  $y$ , and  $E_h$  for expectation under  $P_h$ . The filtering problem discusses

$$E_h[x_t | Y_t],$$

the smoothing problem discusses

$$E_h[x_t | Y_T], \text{ where } t \leq T,$$

and the prediction problem discusses

$$E_h[x_t | Y_T], \text{ where } t \geq T.$$

In this paper, using the techniques of the Malliavin calculus, we show in the filtering, smoothing and prediction cases, that the conditional distribution of  $x_t$  has a smooth density.

#### 4. INTEGRATION BY PARTS

Suppose  $0 < t \leq T$  and let  $U_{0,T}(\tilde{\omega})$  be an  $\tilde{F}_T$  measurable random variable of the form discussed in Lemma 3.4, that is

$$U_{0,T}(\tilde{\omega}) = \int_0^T \gamma_j(s, B_s) dB_s^j \quad (4.1)$$

where  $\gamma_j \in C_b^\infty([0, \infty) \times R^n)$  for  $1 \leq j \leq n$ . Consider the system given by (3.5), (3.7) and (4.1) on  $(\Omega \times \tilde{\Omega}, F \times \tilde{F}, P \times \tilde{P})$ :

$$\begin{aligned} dx_{s,t} &= X_0(x_{s,t}, y_{s,t})dt + X_i(x_{s,t}, y_{s,t})dw_t^i + \tilde{X}_j(s_{s,t}, y_{s,t})dB_t^j \\ dy_{s,t} &= Y_0(y_{s,t})dt + Y_j(y_{s,t})dB_t^j \\ dL_{s,t} &= L_{s,t} h^j(x_{s,t}, y_{s,t})dB_t^j \\ dU_{s,t} &= \gamma_j(t, B_t)dB_t^j \end{aligned} \quad (4.2)$$

Then Theorem 2.5, with  $(x_{s,s}, y_{s,s}, L_{s,s}, U_{s,s}) = (x, y, 1, 0)$ , can be applied to (4.2) and we can consider the associated stochastic flow. Note  $U_{s,t}$  does not involve  $x, y$ , or  $L$ , and if  $U_{0,s} = U$  then

$$U_{0,t} = U + \int_s^t \gamma_j(s, B_s)dB_s^j. \quad (4.3)$$

Also, if  $L = L_{0,s}$ , from (3.8)

$$L_{0,t} = LL_{s,t}(s, y). \quad (4.4)$$

**THEOREM 4.1.** Suppose  $0 < t \leq T$  and let  $c$  be a  $C^\infty$  function on  $R^d$  with bounded derivatives of all orders. Then for any square integrable predictable process  $u(s) =$

$$(u_1(s), \dots, u_m(s))$$

$$\begin{aligned}
& E[U_{0,T} L_{0,T} c(x_{0,t}(x_0, y_0)) \int_0^T u_i(s) dw_s^i] \\
&= \sum_{i=1}^m E[U_{0,T} L_{0,T} c_x(x_{0,t}(x_0, y_0)) D_{0,t} \int_0^t D_{0,s}^{-1} X_i(s) u_i(s) ds] \\
&+ \sum_{i=1}^m E[U_{0,T} L_{0,T} L_{0,T}^{-1} c(x_{0,t}(x_0, y_0)) \frac{\partial L_{0,T}}{\partial x_0} \int_0^T D_{0,s}^{-1} X_i(s) u_i(s) ds] \\
&- \sum_{i=1}^m E[U_{0,T} L_{0,T} c(x_{0,t}(x_0, y_0)) \int_0^T L_{0,s}^{-1} \frac{\partial L_{0,T}}{\partial x_0} D_{0,s}^{-1} X_i(s) u_i(s) ds]. \quad (4.5)
\end{aligned}$$

PROOF. First recall the derivation of Theorem 3.1 and write for  $0 \leq s \leq t \leq T$

$$\begin{aligned}
V(s, x, y, L, U) &= E[U_{0,T} L_{0,T}(x_0, y_0) c(x_{0,t}(x_0, y_0)) | G_s] \\
&= E[(U + U_{s,T}) L L_{s,T}(x, y) c(x_{s,t}(x, y)) | G_s] \\
&= E_{s,x,y,L,U} [(U + U_{s,T}) L L_{s,T}(x, y) c(x_{s,t}(s, y))]. \quad (4.6)
\end{aligned}$$

The martingale representation result is obtained by writing down the Ito formula for  $V$ , and the derivatives of  $V$  are found by differentiating the conditional expectation (4.5) in  $x, y, L$  and  $U$ . Note that for  $s > t$  the derivative of  $c(x_{s,t}(x, y))$  in  $x$  is zero. We,

therefore, have

$$\begin{aligned}
U_{0,T} L_{0,T} (x_0, y_0) c(x_{0,t}(x_0, y_0)) &= E[U_{0,T} L_{0,T} (x_0, y_0) c(x_{0,t}(x_0, y_0))] \\
&+ \int_0^t E[U_{0,T} L_{0,T} c_x(x_{0,t}(x_0, y_0)) D_{0,t} |G_s| D_{0,s}^{-1} (X_i dw_s^i + \tilde{X}_j dB_s^j) \\
&+ \int_0^t E[U_{0,T} L_{0,T} c_x(x_{0,t}(x_0, y_0)) \frac{\partial(x_{0,t}(x, y))}{\partial y} |G_s| Y_j dB_s^j \\
&+ \int_0^T E[U_{0,T} L_{s,T} c(x_{0,t}(x_0, y_0)) |G_s| h^j dB_s^j \\
&+ \int_0^T E[U_{0,T} L \frac{\partial L_{s,T}}{\partial x}(x, y) c(x_{0,t}(x_0, y_0)) |G_s| (X_i dw_s^i + \tilde{X}_j dB_s^j) \\
&+ \int_0^T E[U_{0,T} L \frac{\partial L_{s,T}}{\partial x}(x, y) c(x_{0,t}(x_0, y_0)) |G_s| Y_j dB_s^j \\
&+ \int_0^T E[U_{s,T} L_{0,T} c(x_{0,t}(x_0, y_0)) |G_s| \gamma_j dB_s^j. \tag{4.7}
\end{aligned}$$

Taking the product of (4.7) with  $\int_0^T u_i(s) dw_s^i$ , because  $w$  and  $B$  are independent under  $P \times \tilde{P}$ , we have

$$\begin{aligned}
&E[U_{0,T} L_{0,T} c(x_{0,t}) \int_0^T u_i(s) dw_s^i] \\
&= \sum_{i=1}^m E[U_{0,T} L_{0,T} c_x(x_{0,t}) D_{0,t} \int_0^t D_{0,s}^{-1} X_i(s) u_i(s) ds] \\
&+ \sum_{i=1}^m E[U_{0,T} L c(x_{0,t}) \int_0^T \frac{\partial L_{s,T}}{\partial x}(x, y) X_i(s) ds]. \tag{4.8}
\end{aligned}$$

From (3.9)  $L \frac{\partial L_{s,T}}{\partial x} = \frac{\partial L_{0,T}}{\partial x_0} D_{0,s}^{-1} - \frac{\partial L_{0,s}}{\partial x_0} L_{s,T}(x, y) D_{0,s}^{-1}$ . Substituting in (4.8) the result follows.

NOTATION 4.2. Write  $*$  for the transpose. Furthermore write  $R_{0,T} = \int_0^T (D_{0,s}^{-1} X_i(s))^* dw_s^i$ ,  $\Lambda_{0,T} = \sum_{i=1}^m \int_0^T L_{0,s}^{-1} \frac{\partial L_{0,s}}{\partial x_0} D_{0,s}^{-1} X_i(s) X_i(s)^* (D_{0,s}^{-1})^* ds$  and recall the Malliavin matrix, [1], [4], (which here is a 'partial' Malliavin matrix in the  $X_i$  vector fields):

$$M_{0,T} = \sum_{i=1}^m \int_0^T D_{0,s}^{-1} X_i(s) X_i(s)^* (D_{0,s}^{-1})^* ds.$$

COROLLARY 4.3. We then have the special case of Theorem 4.1 obtained by taking  $u_i(s) = (D_{0,s}^{-1} X_i(s))^*$  :

$$\begin{aligned} E[U_{0,T} L_{0,T} c(x_{0,t}) R_{0,T}] &= E[U_{0,T} L_{0,T} c_x(x_{0,t}) D_{0,t} M_{0,t}] \\ &+ E[U_{0,T} L_{0,T} L_{0,T}^{-1} c(x_{0,t}) \frac{\partial L_{0,T}}{\partial x_0} M_{0,T}] \\ &- E[U_{0,T} L_{0,T} c(x_{0,t}) \Lambda_{0,T}] \end{aligned} \quad (4.9)$$

COROLLARY 4.4. Equation (4.9) is still true for vector, (or matrix), functions  $c$ .

REMARKS 4.5 The gradient  $c_x$  of  $c$  occurs in only one term, so (4.9) is an 'integration by parts' formula. Suppose  $g$  is a second smooth function with bounded derivatives of all orders. Applying (4.9) to the product  $c(x_{0,t})g(x_{0,t})$  we have

$$\begin{aligned} E[U_{0,T} L_{0,T} c(x_{0,t})g(x_{0,t}) R_{0,T}] &= E[U_{0,T} L_{0,T} (c_x(x_{0,t})g(x_{0,t}) + c(x_{0,t})g_x(x_{0,t})) D_{0,t} M_{0,t}] \\ &+ E[U_{0,T} L_{0,T} L_{0,T}^{-1} c(x_{0,t})g(x_{0,t}) \frac{\partial L_{0,T}}{\partial x_0} M_{0,T}] \\ &- E[U_{0,T} L_{0,T} c(x_{0,t})g(x_{0,t}) \Lambda_{0,T}]. \end{aligned} \quad (4.10)$$

From Lemma 3.4 the random variables  $U_{0,T}$  generate  $\tilde{F}_T$  so (4.10) can be written

$$\begin{aligned} E[L_{0,T} c(x_{0,t})g(x_{0,t}) R_{0,T} | \tilde{F}_T] &= E[L_{0,T} (c_x(x_{0,t})g(x_{0,t}) + c(x_{0,t})g_x(x_{0,t})) D_{0,t} M_{0,t} | \tilde{F}_T] \\ &+ E[L_{0,T} L_{0,T}^{-1} c(x_{0,t})g(x_{0,t}) \frac{\partial L_{0,T}}{\partial x_0} M_{0,T} | \tilde{F}_T] \\ &- E[L_{0,T} c(x_{0,t})g(x_{0,t}) \Lambda_{0,T} | \tilde{F}_T]. \end{aligned}$$

Under  $P \times \tilde{P}$ ,  $Y_T \subset \tilde{F}_T$  so

$$\begin{aligned}
& E[L_{0,T} c(x_{0,t}) g(x_{0,t}) R_{0,T} | Y_T] \\
&= E[L_{0,T} (c_z(x_{0,t}) g(x_{0,t}) + c(x_{0,t}) g_z(x_{0,t})) D_{0,t} M_{0,t} | Y_T] \\
&+ E[L_{0,T} L_{0,T}^{-1} c(x_{0,t}) g(x_{0,t}) \frac{\partial L_{0,T}}{\partial x_0} M_{0,T} | Y_T] \\
&- E[L_{0,T} c(x_{0,t}) g(x_{0,t}) \Delta_{0,T} | Y_T].
\end{aligned}$$

Now  $E_h[c(x_{0,t}) g(x_{0,t}) R_{0,T} | Y_T] = E[L_{0,T} c(x_{0,t}) g(x_{0,t}) R_{0,T} | Y_T] (E[L_{0,T} | Y_T])^{-1}$ . Furthermore,  $L_{0,T} > 0$  a.s.; therefore

$$E[L_{0,T} | Y_T]^{-1} < \infty \text{ a.s.}$$

Consequently, dividing by  $E[L_{0,T} | Y_T]$  we have

$$\begin{aligned}
& E_h[c(x_{0,t}) g(x_{0,t}) R_{0,T} | Y_T] \\
&= E_h[(c_z(x_{0,t}) g(x_{0,t}) + c(x_{0,t}) g_z(x_{0,t})) D_{0,t} M_{0,t} | Y_T] \\
&+ E[L_{0,T}^{-1} c(x_{0,t}) g(x_{0,t}) \frac{\partial L_{0,T}}{\partial x_0} M_{0,T} | Y_T] \\
&- E_h[c(x_{0,t}) g(x_{0,t}) \Delta_{0,T} | Y_T], \tag{4.11}
\end{aligned}$$

where both sides are finite a.s.

With this in mind, to obtain a bound for the conditional expectation  $E_h[c_z(x_{0,t}) | Y_T]$  we would like to take  $g = M_{0,t}^{-1} D_{0,t}^{-1}$  in (4.11). However,  $D_{0,t}$  and  $M_{0,t}$  involve the past of the process  $\xi_{0,t}$ ,  $D_{0,t}$  and  $M_{0,t}$ . This difficulty can be circumvented by considering an enlarged system.

NOTATION 4.6. Let  $\phi^{(0)}(\omega, \tilde{\omega}, s, t, x, y, L, U)$  denote the flow associated with the system (4.2). Write  $D_{s,t}^{(0)}$  for the Jacobian associated with this flow. Note that among the components of  $D_{s,t}^{(0)}$  are the 'partial' Jacobian  $\frac{\partial x_{s,t}(x,y)}{\partial x} = D_{s,t}$  and the gradient  $\frac{\partial L_{s,t}(x,y)}{\partial x}$ . Write

$$\begin{aligned} R_{s,t}^{(0)} &= R_{s,t} = \int_s^t (D_{s,u}^{-1} X_i(u))^* dw_u^i \\ \Lambda_{s,t}^{(0)} &= \Lambda_{s,t} = \sum_{i=1}^m \int_s^t L_{s,u}^{-1} \frac{\partial L_{s,u}}{\partial x_s} D_{s,u}^{-1} X_i(u) X_i(u)^* (D_{s,u}^{-1})^* du \\ M_{s,t}^{(0)} &= M_{s,t} = \sum_{i=1}^m \int_s^t D_{s,u}^{-1} X_i(u) X_i(u)^* (D_{s,u}^{-1})^* du. \end{aligned}$$

Then the system

$$\phi^{(1)} = (\phi^{(0)}, D^{(0)}, R^{(0)}, M^{(0)}, \Lambda^{(0)})$$

is Markov with coefficients in

$$S(d+p+2, d+p+2+(d+p+2)^2, 2d+p+2+(d+p+2)^2, 2d+p+2+2(d+p+2)^2, 1).$$

The results of Theorem 2.5 apply to this system and its flow  $\phi^{(1)}$ . Note that  $M_{s,t}$  is the predictable quadratic variation of the tensor product of  $R_{s,t}$  with itself. Write  $X_i^{(1)}$  for the coefficient vector fields of the  $w^i$  integrals in  $\phi^{(1)}$ , and  $D_{s,t}^{(1)}$  for the Jacobian of  $\phi^{(1)}$ . Also write

$$R_{s,t}^{(1)} = \int_0^t (D_{s,u}^{(1)-1} X_i^{(1)}(u)) dw_u^i$$

and  $M_{s,t}^{(1)}$  for the predictable quadratic variation of the tensor product of  $R_{s,t}^{(1)}$  with  $R_{s,t}^{(0)}$  which we shall denote by

$$M_{s,t}^{(1)} = \langle R_{s,t}^{(1)} \otimes R_{s,t}^{(0)} \rangle.$$

Then define  $\phi^{(2)} = (\phi^{(1)}, D^{(1)}, R^{(1)}, M^{(1)})$  so  $\phi^{(2)}$  is a Markov process for which the stochastic flow results of Theorem 2.5 hold. Proceeding in this way we inductively define  $X_i^{(n)}$  for the coefficient vector fields of the  $w^i$  integrals in  $\phi^{(n)}$ ,

$$R_{s,t}^{(n)} = \int_0^t (D_{s,u}^{(n)-1} X_i^{(n)}(u)) dw_u^i,$$

$$M_{s,t}^{(n)} = \langle R^{(n)} \otimes R^{(0)} \rangle_{s,t}$$

and  $\phi^{(n+1)} = (\phi^{(n)}, D^{(n)}, R^{(n)}, M^{(n)})$ . Write  $\nabla_n$  for the gradient operator in the components of  $\phi^{(n)}$ .

**THEOREM 4.7.** *Suppose  $c$  is a bounded  $C^\infty$  scalar function on  $R^d$  with bounded derivatives. Let  $g$  be a possibly vector, (or matrix), valued function on the state space of  $\phi^{(n)}$  such that  $g(\phi^{(n)}(0, t))$  and  $\nabla_n g(\phi^{(n)}(0, t))$  are both in some  $L^p(\Omega)$ . Then*

$$\begin{aligned} & E_h[c(x_{0,t})g(\phi^{(n)}(0, t)) \otimes R_{0,T}^{(0)} | Y_T] \\ &= E_h[c_z(x_{0,t})g(\phi^{(n)}(0, t))D_{0,t}M_{0,t} | Y_T] \\ &+ E_h[c(x_{0,t})\nabla_n g(\phi^{(n)}(0, t))D_{0,t}^{(n)}M_{0,t}^{(n)} | Y_T] \\ &+ E_h[L_{0,T}^{-1}c(x_{0,t})g(\phi^{(n)}(0, t))\frac{\partial L_{0,T}}{\partial x_0}M_{0,T} | Y_T] \\ &- E_h[c(x_{0,t})g(\phi^{(n)}(0, t))\Delta_{0,T} | Y_T]. \end{aligned}$$

**PROOF.** The result follows by applying to the system  $\phi^{(n)}$  the techniques used to derive (4.11).



REMARKS 4.8. Theorem 2.5 implies  $\sup_{s \leq t} |D_{0,s}^{(n)}|$ ,  $\sup_{s \leq t} |M_{0,s}^{(n)}|$ ,  $\sup_{s \leq t} |\frac{\partial L_{0,s}}{\partial x_0}|$ ,  $\sup_{s \leq t} |\Lambda_{0,s}|$  are in  $L^p(\Omega \times \tilde{\Omega}, P \times \tilde{P})$  for all  $1 \leq p < \infty$  and, therefore, finite a.s. . We have already noted that

$$\sup_{s \leq t} |D_{0,s}^{-1}| \text{ and } \sup_{s \leq t} L_{0,s}^{-1}$$

are in every  $L^p(\Omega \times \tilde{\Omega}, P \times \tilde{P})$ ,  $1 \leq p < \infty$ . To write out the above result in terms of  $D_{0,t}$ ,  $\frac{\partial L_{0,t}}{\partial x}$  and higher derivatives involves very involved calculations. Even in the one dimensional case it seems better to introduce the sequence of flows  $\phi^{(n)}$ . Theorem 4.7 can again be thought of as giving a conditional 'integration by parts' formula for  $c_z$ .

COROLLARY 4.9. If  $M_{0,T}^{-1}$  is in some  $L^p(\Omega \times \tilde{\Omega}, P_h)$  taking  $g(\phi^{(1)}(0,t)) = M_{0,t}^{-1} D_{0,t}^{-1}$  in Theorem 4.7 we have

$$\begin{aligned} E_h[c_z(x_{0,t})|Y_T] &= E_h[c(x_{0,t})M_{0,t}^{-1} D_{0,t}^{-1} \otimes R_{0,T}|Y_T] \\ &\quad - E_h[c(x_{0,t})(\nabla_1 g)(D_{0,t}, M_{0,t})D_{0,t}^{(1)} M_{0,t}^{(1)}|Y_T] \\ &\quad - E_h[c(x_{0,t})L_{0,T}^{-1} M_{0,t}^{-1} D_{0,t}^{-1} \frac{\partial L_{0,T}}{\partial x_0} M_{0,T}|Y_T] \\ &\quad + E_h[c(x_{0,t})M_{0,t}^{-1} D_{0,t}^{-1} \Lambda_{0,T}|Y_T]. \end{aligned}$$

Because the remaining terms are integrable and, therefore, finite a.s. we have proved the following result:

THEOREM 4.10. Suppose  $P_h$  is the probability measure of Definition 3.8 and  $(x_{0,t}, y_{0,t})$  is the solution under  $P_h$  of (3.10). Let  $c$  be a smooth function with bounded derivatives of all orders. Then if  $M_{0,T}^{-1}$  is in some  $L^p(\Omega \times \tilde{\Omega}, P_h)$

$$|E[c_z(x_{0,t})|Y_T]| \leq K(y) \sup_{z \in R^d} |c(z)| \quad (4.12)$$

where  $K(y)$  is  $Y_T$ -measurable and finite a.s.

REMARKS 4.11. Condition (4.12) implies that the random variable  $x_{0,t}(x_0, y_0)$  has a conditional density given  $Y_T$ ,  $d(x)$ ,  $x \in R^d$  for almost all  $y$ . (See Malliavin [6] or Zakai [9]). Now for any  $s \leq T$

$$Y_s \subset Y_T.$$

so by Jensen's inequality, from (4.12)

$$|E[c_x(x_{0,t})|Y_s]| \leq K'(y) \sup_{x \in R^d} |c(x)|. \quad (4.13)$$

Equation (4.13) holds for  $s \leq t$  or  $s \geq t$  so the prediction, filtering and smoothing problems for the random variable  $x_{0,t}(x_0, y_0)$  all have a density for almost all  $y$ .

The remaining question concerns the existence and integrability properties of  $M_{0,T}^{-1}$ . These have been carefully studied, see Bismut [1], Malliavin [6] and Stroock [8]. For  $(x, y) \in R^d \times R^p$  write  $T_{x,y}$  for the vector subspace of  $R^d$  generated by the vector fields  $X_1(x, y), \dots, X_m(x, y)$ , and the Lie brackets of  $X_1(x, y), \dots, X_m(x, y)$  and  $\tilde{X}_1(x, y), \dots, \tilde{X}_n(x, y)$ , where each bracket contains at least one of the vector fields  $X_1(x, y), \dots, X_m(x, y)$ . Then in Theorem 1.19 of [2] Bismut and Michel show that for all  $T > 0$ ,  $M_{0,T}^{-1}$  is in  $L^p(\Omega \times \tilde{\Omega}, P_h)$  for all  $1 \leq p < \infty$  if the following condition  $H$  of Hörmander is satisfied:

$$H : T_{x_0, y_0} \text{ is equal to the whole of } R^d.$$

As Bismut [1] observes, if  $H$  is satisfied at  $(x_0, y_0)$  then it is satisfied in some neighbourhood of  $(x_0, y_0)$ . Condition  $H$  is a local statement and translates immediately to diffusions defined on manifolds.

Finally recall that if  $u$  is a non-singular linear map of  $R^d$  to itself, then the map  $\phi : u \rightarrow u^{-1}$  has a derivative  $\phi'(u)$  which is a linear map on the space of linear maps of  $R^d$  to itself given by  $\phi'(u) \cdot h = -u^{-1} \cdot h \cdot u^{-1}$ . Applying this to  $g(D_{0,t}, M_{0,t}) = M_{0,t}^{-1} D_{0,t}^{-1}$  we have

$$\begin{aligned}
E_h[c_z(x_{0,t})|Y_T] &= E_h[c(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1} \otimes R_{0,T}|Y_T] \\
&\quad - E_h[c(x_{0,t})M_{0,t}^{-1}((\nabla_1 M_{0,t})(D_{0,t}^{(1)}M_{0,t}^{(1)})M_{0,t}^{-1}D_{0,t}^{-1})|Y_T] \\
&\quad - E_h[c(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1}((\nabla_1 D_{0,t})(D_{0,t}^{(1)}M_{0,t}^{(1)})D_{0,t}^{-1})|Y_T] \\
&\quad - E_h[c(x_{0,t})L_{0,T}^{-1}M_{0,t}^{-1}D_{0,t}^{-1}\frac{\partial L_{0,T}}{\partial x_0}M_{0,t}|Y_T] \\
&\quad + E_h[c(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1}\Lambda_{0,T}|Y_T].
\end{aligned} \tag{4.14}$$

## 5. BOUNDS FOR HIGHER DERIVATIVES

To show the conditional density of  $x_{0,t}$  is differentiable, in the prediction, filtering and smoothing situations, we shall obtain bounds for higher derivatives of the form:

$$|E[\frac{\partial^\alpha c}{\partial x^\alpha}(x_{0,t})|Y_T]| \leq K(y) \sup_{x \in R^d} |c(x)|. \quad (5.1)$$

where  $0 < t \leq T$ . Here  $\alpha = (\alpha_1, \dots, \alpha_d)$  is a multi-index of non-negative integers and

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}}.$$

Again if  $0 \leq s \leq T$  then Jensen's inequality applied to (5.1) gives

$$|E[\frac{\partial^\alpha c}{\partial x^\alpha}(x_{0,t})|Y_s]| \leq K'(y) \sup_{x \in R^d} |c(x)|. \quad (5.2)$$

A well known argument from harmonic analysis, (see [7], or [8]), shows that if (5.2) is true for all  $\alpha$  with  $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d \leq n$  where  $n \geq d+1$  then the random variable  $x_{0,t}(x_0, y_0)$  has a conditional density  $d(x)$  given  $Y_s$ , which is in  $C^{n-d-1}(R^d)$ . That is, we have a differentiable conditional density in the prediction, filtering and smoothing situations.

To see how to proceed apply Corollary 4.9 to  $c_x$  in place of  $c$ . (If preferred, Corollary 4.9 could be applied to just one partial derivative  $\frac{\partial c}{\partial x_k}$  in place of  $c$ . However, the result is true for vector functions  $c$ ). This gives:

$$\begin{aligned} E_h[c_{xx}(x_{0,t})|Y_T] &= E_h[c_x(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1} \otimes R_{0,T}|Y_T] \\ &\quad - E_h[c_x(x_{0,t})(\nabla_1 g)(D_{0,t}, M_{0,t})D_{0,t}^{(1)}M_{0,t}^{(1)}|Y_T] \\ &\quad - E_h[c_x(x_{0,t})L_{0,T}^{-1}M_{0,t}^{-1}D_{0,t}^{-1}\frac{\partial L_{0,T}}{\partial x_0}M_{0,T}|Y_T] \\ &\quad + E_h[c_x(x_{0,t})M_{0,t}^{-1}D_{0,t}^{-1}\Lambda_{0,T}|Y_T]. \end{aligned} \quad (5.3)$$

Consider the four terms on the right of (5.3). Each term is of the form

$$E_h[c_x(x_{0,t})h_i(\phi^{(1)}(0,t), \phi^{(1)}(0,T))|Y_T], i = 1, 2, 3, 4.$$

For each such  $h_i$  consider a function

$$\tilde{h}_i = h_i M_{0,t}^{-1} D_{0,t}^{-1}$$

and apply Theorem 4.7 to  $c$  and  $\tilde{h}_i$  to obtain

$$\begin{aligned} & E_h[c_x(x_{0,t})h_i(\phi^{(1)}(0,t), \phi^{(1)}(0,T))|Y_T] \\ &= E_h[c(x_{0,t})\tilde{h}_i(\phi^{(1)}(0,t), \phi^{(1)}(0,T)) \otimes R_{0,T}^{(0)} |Y_T] \\ &- E_h[c(x_{0,t})(\nabla_n(t)\tilde{h}_i)(\phi^{(1)}(0,t), \phi^{(1)}(0,T))D_{0,t}^{(2)} M_{0,t}^{(2)} |Y_T] \\ &- E_h[c(x_{0,t})(\nabla_n(T)\tilde{h}_i)(\phi^{(1)}(0,t), \phi^{(1)}(0,T))D_{0,T}^{(2)} M_{0,T}^{(2)} |Y_T] \\ &- E_h[L_{0,T}^{-1} c(x_{0,t})\tilde{h}_i(\phi^{(1)}(0,t), \phi^{(1)}(0,T)) \frac{\partial L_{0,T}}{\partial x_0} M_{0,T} |Y_T] \\ &+ E_h[c(x_{0,t})\tilde{h}_i(\phi^{(1)}(0,t), \phi^{(1)}(0,T))\Delta_{0,T} |Y_T]. \end{aligned}$$

Substituting in (5.3) we obtain an expression on the right which involves only  $c$  and not its derivatives. This procedure can be repeated, using Theorem 4.7. At each stage, to replace a term of the form  $E_h[c_x(x_{0,t})h(\phi^{(n)}(0,t), \phi^{(n)}(0,T))|Y_T]$  by one involving only  $c$  define  $\tilde{h} = h M_{0,t}^{-1} D_{0,t}^{-1}$  and applying Theorem 4.7. Clearly higher powers of  $M_{0,t}^{-1}$  are introduced at each iteration. However, Hörmander's condition  $H$  is sufficient to ensure that  $M_{0,t}^{-1}$  is in every  $L^p(\Omega \times \tilde{\Omega}, P_h)$ ,  $1 \leq p < \infty$ . We have, therefore, proved the following result:

THEOREM 5.1. Suppose Hörmander's condition  $H$  is satisfied. Then the random variable  $x_{0,t}(x_0, y_0)$ , the solution of the signal process, has a conditional density given  $Y_s$  for almost all  $y$  which is in  $C^\infty(R^d)$  for  $s \geq t$  and  $s \leq t$ . That is, under condition  $H$  the prediction, filtering and smoothing problems have a smooth density solution.

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